Info Quantique ENSEA - Exercises

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February 4, 2016

1 Un peu d'algèbre

Dans ce qui suit, $\{|0\rangle, |1\rangle\}$ est la *base de calcul* ("computational basis", terme consacré). C'est une base orthonormée qui engendre l'espace de Hilbert $\mathbb{C}^2$. On note $\{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$ la base de Hadamard (qui peut être obtenue en appliquant une transformation de Hadamard à la base de calcul). La notation $|xyz\rangle$ est un "raccourci" pour $|x\rangle \otimes |y\rangle \otimes |z\rangle$.

Q1. Quelle est la représentation matricielle de l'opérateur $|0\rangle\langle 0| + |1\rangle\langle 1|$: i) dans la base de calcul ii) dans la base de Hadamard ?

Q2. L'opérateur NOT est défini par la table de vérité $|0\rangle \to |1\rangle$ et $|1\rangle \to |0\rangle$. Quel opérateur unitaire implémente cette opération ?

Q3. Soit $\mathbf{n}$ un vecteur unitaire de $\mathbb{R}^3$, $\{\mathbf{1}, X, Y, Z\}$ le groupe de Pauli, et on considère l'opérateur linéaire

$$\Pi(\mathbf{n}) = \frac{1}{2}(\mathbf{1} + \mathbf{n} \cdot \sigma)$$

où $\mathbf{n} \cdot \sigma = n_x X + n_y Y + n_z Z$. Trouver $\Pi^\dagger(\mathbf{n})$, $\mathrm{tr}(\Pi(\mathbf{n})$ et $\Pi^2(\mathbf{n})$. Calculer $\exp(i\theta\mathbf{n} \cdot \sigma)$ où $\theta \in \mathbb{R}$.

Q4. Comparer les produits tensoriels d'opérateurs suivants : $X \otimes Z$ et $Z \otimes X$.

Q5. Est-ce que l'état $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ peut être factorisable en un produit de deux états ? Et $|\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle$?

Q6. La transformation de Hadamard s'écrit

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Soit $W = H \otimes H$. Déterminer $W|00\rangle$.

Q7. Montrer que $\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B)$ pour tout opérateur A et B.

Q8. Le groupe de Lie $SU(2)$ est défini comme le groupe des matrices unitaires 2×2 à déterminant unité et à valeurs complexes. Montrer que les membres de $SU(2)$ peuvent s'écrire $x_0\mathbf{1} + i\mathbf{x} \cdot \sigma, x_0^2 + \mathbf{x}^2 = 1$

2 Density matrices for simple systems

Let $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$.

Q9. Calculate the corresponding density matrix.

Q10. Check whether ρ corresponds to a pure state or not.

In this question, we will show that pure states are located on the boundary of the Bloch ball, whereas mixed state are inside the ball. Let ρ be the density matrix for a mixed state.

Q11. Show that ρ can be written as $a\mathbf{1} + bX + cY + dZ$ where (a, b, c, d) are reals. What is the condition on these numbers?

Q12. Show that $\rho = \frac{1}{2}(\mathbf{1} + xX + yY + zZ)$ where (x, y, z) are the cartesian coordinates of some vector of the three dimensional euclidean space.

Q13. Calculate $\det\rho$ for a pure state and for a mixed state. Discuss with respect to the Bloch sphere.

Let now consider a mixed state that corresponds to a mixture of qu-bits pointing anywhere with equal probability (that is, any point on the Bloch sphere has equal weight).

Q14. Calculate the density matrix of such a state. Check that this is not a pure state indeed.

Q15. Compute the expectation value of X, Y and Z.

In the following we examine the polarization of a qu-bit. Let $\rho = \frac{1}{2}(\mathbf{1} + \mathbf{r}\cdot\sigma)$ where $\mathbf{r} = (x, y, z)$ is a vector of the Bloch ball and $\sigma = (X, Y, Z)$.

Q16. Let $\mathbf{n}$ be a unitary vector. Calculate the expectation value of the observable $\sigma_{\mathbf{n}}$ that measures the component of the qu-bit along the $\mathbf{n}$ axis. In what respect does $\mathbf{r}$ tell the degree of polarization of the qu-bit?

Non-uniqueness of decomposition of ρ:

Q17. Consider the following mixtures:

- a mixture of two third of state $|0\rangle$ and one third of state $|1\rangle$.
- a mixture of 50% of state $|a\rangle = \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle$ and 50% of state $|b\rangle = \sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{1}{3}}|1\rangle$.

Are the density matrices the same?

Q18. Show that two density matrices for a qu-bit commute if their Bloch vectors are parallel.

Q19. Let $r \in [0, 1]$. Consider the density matrix

$$\rho = r\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + (1-r)|00\rangle\langle 00|$$

where $|\Phi^+\rangle$ is the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Calculate the eigenvalues of ρ.

3 Density matrices for composite systems

Q20. Assume that the density matrix for a given composite system AB is separable, i.e., $\rho^{AB} = \rho^A \otimes \rho^B$. Calculate the reduced density matrix for each subsystem A and B.

4 Entropy

Q21. Consider the density matrix

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Calculate the entropy of information of the system defined by $S(\rho) = -\operatorname{tr} \rho \log_2 \rho$.

Q22. Let ρ^{AB} be a density matrix defined on a $N \times N$-dimensional Hilbert space $H \otimes H$. The *classical information capacity* is defined as

$$C(\rho) = \log_2 N + S(\rho^B) - S(\rho^{AB})$$

Consider the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \right)$$

Calculate ρ^{AB}, the reduced density matrix ρ^B and then the classical information capacity.

Q23. The *quantum relative entropy* between two density operators ρ and σ is defined by

$$S_b(\rho||\sigma) = \operatorname{tr}(\rho \log_b \rho - \rho \log_b \sigma) = -S_b(\rho) - \operatorname{tr} \rho \log_b \sigma$$

Here S_b is the Van Neumann entropy where the log is taken with the base b. Show that $S_b(\rho||\sigma) \geq 0$. This inequality is know as Klein's inequality.

5 Measurement

A positive operator-valued measure (POVM) is a collection

$$\{E_j : j = 1, 2, \ldots, n\}$$

of nonnegative (positive semi-definite) operators, satisfying

$$\sum_{j=1}^{n} = \mathbf{1}$$

In other words a partition of unity by nonnegative operators is called a positive operator-valued measure (POVM). When a state $|\psi\rangle$ is subjected to such a POVM, outcome j occurs with probability

$$p(j) = \langle \psi | E_j | \psi \rangle$$

Consider a qu-bit system. Let

$$E_1 = |0\rangle\langle 0|, E_2 = |1\rangle\langle 1|$$

and

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Q24. Find $p(1)$ and $p(2)$.

6 Entanglement

One particularly interesting state in quantum computing is the Greenberger-Horne-Zeilinger state (GHZ state). This state of three qubits acts in the Hilbert space $\mathbb{C}^8$ and is given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}[101010 + 010101]$$

Q25. Find the density matrix $\rho = |\psi\rangle\langle\psi|$.

Q26. Show that ρ can be written as a linear combination in terms of tensor products of Pauli matrices.

Let $|\psi\rangle$ be a given state in the Hilbert space $\mathbb{C}^n$. Let X and Y be two $n \times n$ hermitian matrices. We define the correlation for a given state $|\psi\rangle$ as

$$C_{XY}(|\psi\rangle) = \langle XY\rangle - \langle X\rangle\langle Y\rangle$$

Let

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and consider the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}1001$$

Q27. Find the correlation.