Introduction to dynamical systems

One-dimensional iterated maps

September 17, 2015

The logistic map and other maps Let $f(x) = ax(1-x)$ the logistic map.

Q1. Find the period-two orbit for $a = 4$. Are they sinks or sources? Show that $h = fofof$ has eight fixed points, two of which are fixed points of f. Show that the other six fixed points make up two period-three orbits of f (you'll need a computer).

Q2. **Computer experiment**: investigate the long-run behavior of the logistic map for a near $1+\sqrt{6}$, and for a slightly smaller. Discuss the speed of convergence.

Q3. Construct the periodic table of f for $a = 4$:

Period k	Nb of fixed pts of G^k	Nb of fixed points ofG^k due to lower period orbits	Orbits of period k
1	2	0	2
2	4	2	1
3	8	2	2
4	16	4	3
⋮	⋮	⋮	⋮

Discuss the following assertion: the logistic map for $a = 4$ has orbits of every period.

Q4. Let $f(x) = x^2 + x$. Find all fixed points of f.

Q5. Let the iterated map $x_{n+1} = x_n^2 + c$. Find and classify its fixed points (stable/unstable/period-k cycles/...). Find the values of c for which there are bifurcations, and classify these bifurcations. For which value of c is there a stable period-2 cycle? Draw the bifurcation diagram.

Sensitive dependence on initial conditions Consider the map $f(x) = 3x$ mod 1 on the unit interval. Sketch it. Notice that f can be viewed as a map on the circle of circumference one, in which case it becomes continuous (aka differomorphism). We call a point x *eventually periodic* with period p for the map f if for some positive integer N, we have $f^{n+p}(x) = f^n(x)$ for all $n \geq N$, and if p is the smallest such positive integer. For example, $x = 1/3$ is eventually periodic.

A point x_0 has sensitive dependence on initial conditions if there is a nonzero distance d such that some points arbitrarily near x_0 are eventually mapped at least d units from the corresponding image of x_0. More precisely, there exists $d > 0$ such that any neighborhood N of x_0 contains a point x such that $|f^k(x) - f^k(x_0)| > d$ for some nonnegative integer k.

Q6. Show that a point x is eventually periodic if it is a rational number.

Q7. Construct the periodic table of f.

Q8. Computer experiment: compare the orbits of two nearly equal initial conditions: 0.25 and 0.2501.

Q9. Show that the distance between any pair of points that lie within $1/6$ of one another is tripled by the map. Find a pair of points whose distance is not tripled by the map. Show that to prove sensitive dependence for any point, d can be taken to be any positive number less than $1/2$ in the definition above , and that k can be chosen to be the smallest integer greater than $\ln(d/|x - x_0|)/\ln 3$.