# Introduction to dynamical systems

## Two-dimensional iterated maps (1): saddles and basins of attraction.

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Let the Henon map: $f(x,y) = (a - x^2 + by, x)$.

**Q1.** (a) If $(x_1, y_1)$ and $(x_2, y_2)$ are the two fixed points of the Hénon map with some fixed parameters $a$ and $b$, show that $x_1 - y_1 = x_2 - y_2 = 0$ and $x_1 + x_2 = y_1 + y_2 = b - 1$. (b) If $\{(x1, y1), (x2, y2)\}$ is the period-two orbit, show that $x_1 + y_1 = x_2 + y_2 = x_1 + x_2 = y_1 + y_2 = 1 - b$. In particular the period-two orbit lies along the line $x + y = 1 - b$. Prove that the Henon map has a period-two orbit if and only if $4a > 3(1 - b)^2$.

**Q2.** Set $b = 0.4$. Prove that for $-0.09 < a < 0.27$, the Hénon map $f$ has one sink fixed point and one saddle fixed point. Find the largest magnitude eigenvalue of the Jacobian matrix at the first fixed point when $a = 0.27$. Explain the loss of stability of the sink. Prove that for $0.27 < a < 0.85$, $f$ has a period-two sink. What kind of bifurcation(s) do we observe there? Finally, find the largest magnitude eigenvalue of $Df^2$, the Jacobian of $f^2$ at the period-two orbit, when $a = 0.85$.

**Q3.** Computer experiment : compute and then sketch graphically the shape of the attractors for the Henon map with $b = 0.4$, and (a) $a = 0.9$, (b) $a = 0.988$, (c) $a = 1.0$, (d) $a = 1.0293$, (e) $a = 1.045$ and finally (f) $a = 1.2$.

**Q4.** Computer experiment : Compute the basins of attraction of the Henon map for $a = 0$ and $b = 0.4$ (you should find a smooth boundary). Same for $a = 2$ and $b = -0.3$ (now the boundary is more complex).

**Q5.** Computer experiment : for $a = 0$ and $b = 0.4$, consider a unit disk located around each fixed points, and plot a couple of iterates of it.

The next three questions are about other maps...

**Q6.** Let $g(x,y) = (x^2 - 5x + y, x^2)$. Find and classify the fixed points of $g$ as sinks, sources, or saddles.

**Q7.** Let $f(x,y) = (\sin(\pi x/3), y)$. Find all fixed points and their stability. Where does the orbit of each initial value go?

**Q8.** Let $f(x,y) = (x/2, 2y - 7x^2)$. Verify that $(x, 4x^2) : x \in \mathbb{R}$ is the stable manifold of the fixed point $(0,0)$.