Introduction to dynamical systems

ODE's ; the sine-map and the devil's staircase

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1 Pointcaré map from a 2D flow

We consider the ODE system

$$\dot{r} = br(1 - r) \quad (1)$$

$$\dot{\theta} = 1 \quad (2)$$

where b is a positive constant and (r, θ) are polar coordinates. From the corresponding two dimensional flow we wish to compute a (one-dimensional) Poincaré iterated map.

Q1. Find the fixed points and/or limit cycles of the flow.

Q2. Compute the jacobian matrix of the ODE system. What is the stability of limit cycles and/or fixed points?

In order to compute the Poincaré map of the flow, we slice the flow using a half-line denoted as L, starting at the origin of the coordinate system, and forming an arbitrary angle $\bar{\theta}$ with the horizontal axis (cf. Fig. 1). Let $T : r_n \to r_{n+1}$ be the corresponding map from L onto itself. A given trajectory returns on L after a 2π rotation of the flow.

Q3. Find T by integrating the flow from r_n to r_{n+1}.

Q4. From the expression of T, find the value of b for which trajectories converge to a limit cycle ; compute the associated Floquet multipliers (i.e., the derivatives of T at the corresponding fixed point) and check the stability against b.

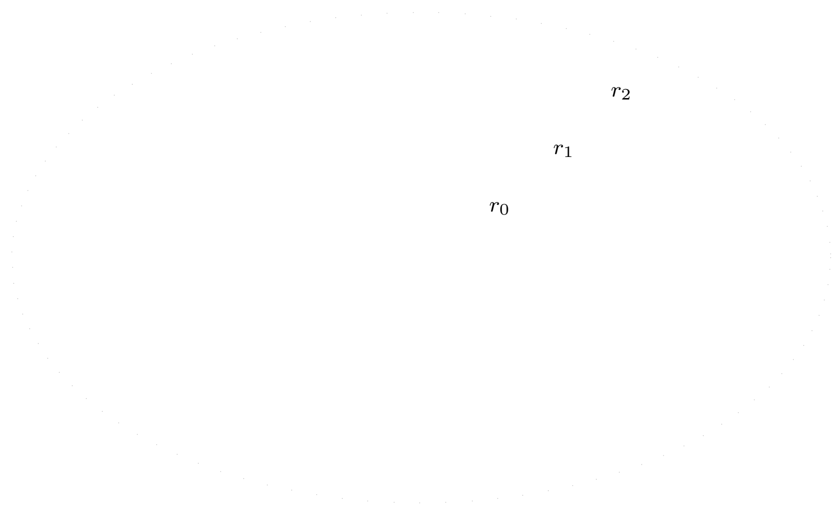

Figure 1: Poincaré map of a two dimensional flow

2 Synchronization of oscillators - the sine-map

Let a system whose dynamics in a Poincaré section of the flow is given by the following iterated map,

$$\theta_{n+1} = f(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_n)$$

where Ω and K are two control parameters (with K measuring the strength of the non-linearity and Ω being the frequency ratio parameter), and θ refers to a linear coordinate along a circle of perimeter 1 (or, stated differently, θ is a coordinate inside the $[0, 1]$ segment, but we let the segment wrap around so that the 0 and 1 points coincide).

We denote the winding number of this map as

$$w = \lim_{n\to\infty} \frac{f^{(n)}(\theta_0) - \theta_0}{n}$$

Q5. Compute w for the sine-map with $K = 0$ (also called the linear-map). We call this number the bare winding number (since $K = 0$).

Q6. For which value of K is the map non-invertible ?

Q7. For what combination of K and Ω are there fixed points ? Compute the winding number for every combination that leads to stable fixed points. Hint: you may first want to examine what happens when $K < 1$. You may also want to pay attention to the fact that angles are computed modulo 1, hence in the general case the fixed point equation shall be written $f(\theta) = \theta + m$ where m is an integer.

Q8. On a graph with Ω and K on the x- and y-axis respectively, sketch (by shading them) the so-called Arnold's tongues, that is, the regions where frequency locking (with small p's and q-'s) occurs. Start with 0:1, 1:1 and 1:2. What could happen above the line $K = 1$?

Q9. Write a computer program that can compute w for $K = 1$ as a function of $0 < \Omega < 1$. Show that this graph displays plateaus over which the winding number does not change for a substantial change of Ω (these are called the Devil's Staircase).