Introduction to dynamical systems

Lyapunov exponents and chaos

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Q1. Find the Lyapunov exponent shared by most bounded orbits of $g(x) = 2.5x(1-x)$. Begin by sketching g(x) and considering the graphical representation of orbits. What are the possible bounded asymptotic behaviors? Do all bounded orbits have the same Lyapunov exponents?

Q2. Write a program to calculate the Lyapunov exponent of $g_a(x) = ax(1-x)$ for values of the parameter a between 2 and 4. Graph the results as a function of a.

Q3. Let $f(x) = (x+q)(mod 1)$, where q is irrational. Check that f has no periodic orbits and that the Lyapunov exponent of each orbit is 0.

Q4. Let the tent map $T(x) = 2x$ if $x < 1/2, f(x) = 2(1-x)$ otherwise . Show that the set of points with itinerary $S_1, ..., S_k$ has length 2^{-k}, independent of the choice of symbols. Explain why each infinite itinerary of the tent map T represents the orbit of exactly one point in $[0, 1]$.

We now examine the link between the properties of the tent map and those of the logistic map.

Q5. Show that the logistic map G and the tent map T are conjugate by the one-to-one continuous map $C(x) = (1-\cos\pi x)/2$.

Q6. Show that if x is a period-k point for T, then $C(x)$ is a period-k point for G. Show that they have the same stability. Use this result to prove that all periodic points of the logistic map G (with $a = 4$) are sources.

Q7. Using conjugacy, show that subintervals of level k of the logistic map have length at most $\pi/2^{k+1}$.

Q8. Consider an orbit $\{x_i\}$ of T that does not contain the point 0. Show that the Lyapunov exponents of the corresponding orbits of T and G are identical.

Q9. Finally, use all these results to show that the logistic map G has chaotic orbits.

Q10. Find a conjugacy C between $G(x) = 4x(1-x)$ and $g(x) = 2-x^2$. Show that $g(x)$ has chaotic orbits.
