Dissipative Quantum and Classical Liouville Mechanics of the Anharmonic Oscillator

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We present a solution for the dynamics of an anharmonic oscillator coupled to a zero-temperature heat bath. Comparison of observable properties in a classical and quantum description uses true joint phase-space probability densities. The time evolution of the density in the quantum case is rapidly "reduced" to that given in the classical description. The rate of reduction is proportional to the product of the damping rate and the oscillator's initial energy. Quite rapidly, typical quantum recurrence effects are destroyed and the classical "whorl" structure restored. We point out the close similarity with rapid destruction of quantum coherence through dissipation.

A dynamical phase-space picture requires the simultaneous specification of position and momentum at varying time instants. Such measurements made upon an ensemble of identically prepared systems yield joint phase-space probability densities. In a quantum description, it is well known that the possible joint probability densities which arise are restricted to a subset of those occurring in a classical description. In a previous paper[1] one of us investigated the quantum and classical dynamics of joint phase-space probability densities describing simultaneous measurements of position and momentum for a particular anharmonic oscillator model with the Hamiltonian

$$H = H_0 + (\mu/\hbar\omega_0)H_0^2, \tag{1}$$

where H_0 is the free Hamiltonian of the simple harmonic oscillator. The appearance of $\hbar$ in Eq. (1) is simply to provide a convenient energy scale; H is in units of $\hbar\omega_0$ when H_0 is in units of $\hbar\omega_0$.

When the system is described classically, an initial Gaussian joint density displaced from the origin develops into a "whorl"[2]; contours of the initial density undergo a rotational shear and as time proceeds spiral out from the origin. However, when the same system is described quantum mechanically the density undergoes a more complicated evolution; "interference" fringes develop, the initial state recurs at a fixed period, and no whorl develops. This more complicated behavior is manifested in the density evolution equation by the appearance of second-order derivatives with complex coefficients. Similar behavior for a related anharmonic model has recently been reported by Takahashi and Saito.[3]

In this Letter we consider the effect of dissipation on the evolution of phase-space densities for a particular anharmonic oscillator model. We show that quite apart from the overall contraction of phase space, the effect of dissipation is to destroy the interference

terms, prevent a recurrence of the initial state and to restore the classical whorl structure. The destruction of interference effects becomes much more rapid as the average energy of the initial state increases (i.e., when the initial density is concentrated at a large radius). This dependence on the initial energy is similar to the decay of off-diagonal coherence in the harmonic oscillator.[4] We are thus lead to interpret the interference terms as a manifestation of quantum coherences, between parts of the density wrapped on neighboring tori.

We incorporate dissipation in our model by coupling the anharmonic oscillator to a reservoir of oscillators which we assume to be at zero temperature. The reservoir is then eliminated and a Markovian master equation for the oscillator-density operator in the interaction picture is obtained. A unique phase-space density is associated with the density operator by means of a bounded positive map from the state space of density operators to the classical state space of probability densities on phase space. This map is then used to transform the evolution equation for the density operator to an evolution equation for the density on a two-dimensional phase space.

The Hamiltonian for the coupled oscillator-reservoir system is

$$H = H_0 + (\mu/\hbar\omega_0)H_0^2 + H_I + H_R, \tag{2}$$

where H_R is the free Hamiltonian for the reservoir and H_I is the oscillator-reservoir interaction Hamiltonian. Following Agarwal[5] we assume H_I to be of the form of position-position coupling and we make the rotating-wave approximation. Thus

$$H_I = \sum_i g_i a b_i^\dagger + g_i^* a^\dagger b_i, \tag{3}$$

where

$$a = (\omega_0/2\hbar)^{1/2}\hat{q} + i(2\hbar\omega_0)^{-1/2}\hat{p}, \tag{4}$$

$$b_i = (\omega_i/2\hbar)^{1/2}\hat{Q}_i + i(2\hbar\omega_i)^{-1/2}\hat{P}_i. \tag{5}$$

a and b_l are the annihilation operators for the oscillator and reservoir, respectively. By use of standard techniques[6] the following master equation is obtained in the Markov approximation and in the interaction picture:

$$\partial\rho/\partial\tau = -i\mu[(a^\dagger a)^2,\rho] + (\gamma/2)[a\rho,a^\dagger] + (\gamma/2)[a,\rho a^\dagger],$$ (6)

where $\tau = \omega_0 t$.

Classical states are represented by joint probability densities on phase space $Q(\alpha,\alpha^*,\tau)$. [We choose to work with the complex parameters α defined as

$$\alpha = (\omega_0/2\hbar)^{1/2}q + i(2\hbar\omega_0)^{-1/2}p,$$

where (q,p) is a point in the phase space of the oscillator.] Quantum states are represented by density operators on Hilbert space. However, there exists a bounded positive operator on quantum state space which maps every quantum state to a unique element in a subset of classical state space. The map $\hat{T}_\alpha$ is defined by

$$\hat{T}_\alpha\rho(t) \equiv \text{Tr}[\rho(t)|\alpha\rangle\langle\alpha|] = Q(\alpha,\alpha^*;t),$$ (7)

where $|\alpha\rangle$ is a minimum-uncertainty state. The resulting joint probability density describes the simultaneous measurements of "approximate" canonical variables q and p relative to some apparatus specified by the choice of the state $|\alpha\rangle$.[7] More generally, $|\alpha\rangle\langle\alpha|d^2\alpha$ is an example of an effect-valued measure.[8] In fact, the effect $|\alpha\rangle\langle\alpha|$ is a special example of a more general effect for simultaneous position and momentum measurements.[9] It corresponds to a situation where the measuring instrument is operated at zero temperature and thus adds no excess noise to the measurement.

Under the map $\hat{T}_\alpha$ the evolution equation in quantum state space becomes an evolution equation in classical state space. For the evolution equation (6) we have

$$\frac{\partial Q}{\partial\tau}(\alpha,\alpha^*;\tau) = \{[\partial_\alpha(\gamma/2 + i\mu(1+2|\alpha|^2)\alpha) + \text{c.c.}] + i\mu\alpha^2\partial_\alpha^2 - i\mu\alpha^{*2}\partial_{\alpha^*}^2 + \gamma\partial_\alpha\partial_{\alpha^*}\}Q(\alpha,\alpha^*;\tau),$$ (8)

where $\partial_\alpha \equiv \partial/\partial\alpha$. When $\gamma = 0$ Eq. (8) reduces to the classical Liouville equation as $\hbar \to 0$.[1]

The evolution equation (8) is solved subject to the initial condition

$$Q(\alpha,\alpha^*;0) = \exp[-|\alpha-\alpha_0|^2].$$ (9)

With respect to the measurements defined by $\hat{T}_\alpha$ discussed above, this density corresponds to the quantum state $\rho = |\alpha_0\rangle\langle\alpha_0|$ where $|\alpha_0\rangle$ is a particular coherent minimum-uncertainty oscillator state. The solution is

$$Q(\alpha,\alpha^*;\tau) = \exp(-|\alpha|^2) \sum_{q,p=0}^{\infty} (q!p!)^{-1}(\alpha\alpha_0^*)^q(\alpha^*\alpha_0)^p f(t)^{(p+q)/2}\exp\{-|\alpha_0|^2[f(t)+i\delta]/(1+i\delta)\},$$ (10)

where

$$\delta = (p-q)/\kappa,$$ (11)

$$f(t) = \exp[-\kappa\nu - i\nu(p-q)],$$ (12)

and

$$\nu = 2\mu\tau,$$ (13)

$$\kappa = (\gamma/2\mu).$$ (14)

The method of solution is similar to that discussed in (1).

In Figs. 1(a)–1(c) we have plotted $Q(\alpha,\alpha^*;\tau)$ for various values of ν, $|\alpha_0|^2$, and κ. When $\gamma = 0$, $Q(\alpha,\alpha^*;\tau)$ exhibits the complicated recurrence behavior discussed in (1). The initial Gaussian starts to form a whorl; however, as the leading edge of the whorl begins to encircle the trailing tail concentrated on an interior torus, "interference fringes" begin to develop. This is evident in Fig. 1(a). These interference fringes become more evident as the evolution

proceeds and eventually result in a complete recurrence of the initial state at $\nu = 2\pi$ (up to a phase of $e^{i\pi}$). This is in distinct contrast to the evolution of a similar density in a classical description, Fig. 1(c); no "interference" fringes arise, no recurrence occurs, and the density becomes concentrated on a "thin" spiral.

However, when $\gamma \neq 0$, but small, we see a significant change in the evolution of the density. As shown in Fig. 1(b) the interference fringes are suppressed and the density becomes similar to its classical analog, Fig. 1(c). However, it is prevented from becoming too narrowly concentrated on a phase-space spiral through the bounded nature of the map $\hat{T}_\alpha$. In fact, for $|\alpha_0| \gg 1$ a very small value of γ is sufficient to suppress the interference features, which decay on a time scale of $(|\alpha_0|^2\kappa)^{-1}$, much shorter than the time scale of the overall contractive dynamics κ^{-1}.

This behavior is reflected in the moments. For example, one may show that

$$\langle a(t)\rangle = \alpha_0\exp[-i\nu(1-i\kappa)/2 - |\alpha_0|^2(1-e^{-i\nu(1-i\kappa)})/(1-i\kappa)].$$ (15)

For $\nu\kappa \ll 1$ and $\kappa \ll 1$, this may be written

$$\langle a(t)\rangle \simeq \alpha_0 e^{-i\nu/2}\exp[-|\alpha_0|^2(1-e^{-i\nu})]\exp[-|\alpha_0|^2\nu\kappa e^{-i\nu}]. \tag{16}$$

For $\kappa = 0 \langle a(t)\rangle$ exhibits the typical periodicity expected; however, for $\kappa \neq 0$ there is a rapid collapse of the recurrence on a time scale $(|\alpha_0|^2\kappa)^{-1}$.

Such a decay rate is quite similar to the decay of off-diagonal coherence in general quantum systems.[4] For example, a damped simple harmonic oscillator initially in the superposition state

$$\rho(0) = N(|\alpha\rangle + |\beta\rangle)(\langle\alpha| + \langle\beta|) \tag{17}$$

evolves to

$$\rho(t) = N \sum_{\gamma,\gamma'=\alpha}^{\beta} \langle\gamma|\gamma'\rangle^{(1-e^{-\nu\kappa})}|\gamma e^{-\nu\kappa/2}\rangle \langle\gamma' e^{-\nu\kappa/2}|. \tag{18}$$

The off-diagonal elements decay via the factor

$$\langle\alpha|\beta\rangle^{(1-e^{-\nu\kappa})} = \exp\{-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\alpha^*\beta)\nu\kappa\}$$

for small $\nu\kappa$. If α and β lie along a common radius we put $\alpha = xe^{i\theta}$ and $\beta = (x+\delta x)e^{i\theta}$ and the decay factor becomes

$$\langle\alpha|\beta\rangle^{(1-e^{-\nu\kappa})} \simeq \exp\{-\delta x^2\nu\kappa/2\}, \tag{19}$$

which is small for large δx. It thus seems reasonable to suggest that the interference terms discussed above arise from off-diagonal coherence due to the density becoming concentrated on adjacent tori. The off-diagonal coherence and consequently the interference are suppressed by dissipation at a rate proportional to the radii of the tori on which the density becomes concentrated. The sizes of the radii are determined by the average energy in the initial state.

There are a number of ways to interpret the dissipation discussed in this Letter. In the first instance, one may claim that the inclusion of dissipation models the effect of connecting an actual measuring instrument to the system, which generates the observed phase-space densities. Interpreting a measuring device as essentially a reservoir from the point of view of the measured system to which it is coupled has been discussed in a number of recent papers.[10-12] The subsequent irreversible interaction ensures that the state of the system becomes diagonal in a unique basis. It thus appears that the very act of observing the dynamic properties of a quantum system is sufficient to produce a rapid approach of the observed behavior to that expected from a classical model.

Of course, quite apart from whether or not the dissipation is attributed to the act of measurement, one may claim that no system is truly isolated and some dissipation, no matter how small, should be included. For example, the anharmonic oscillator discussed here
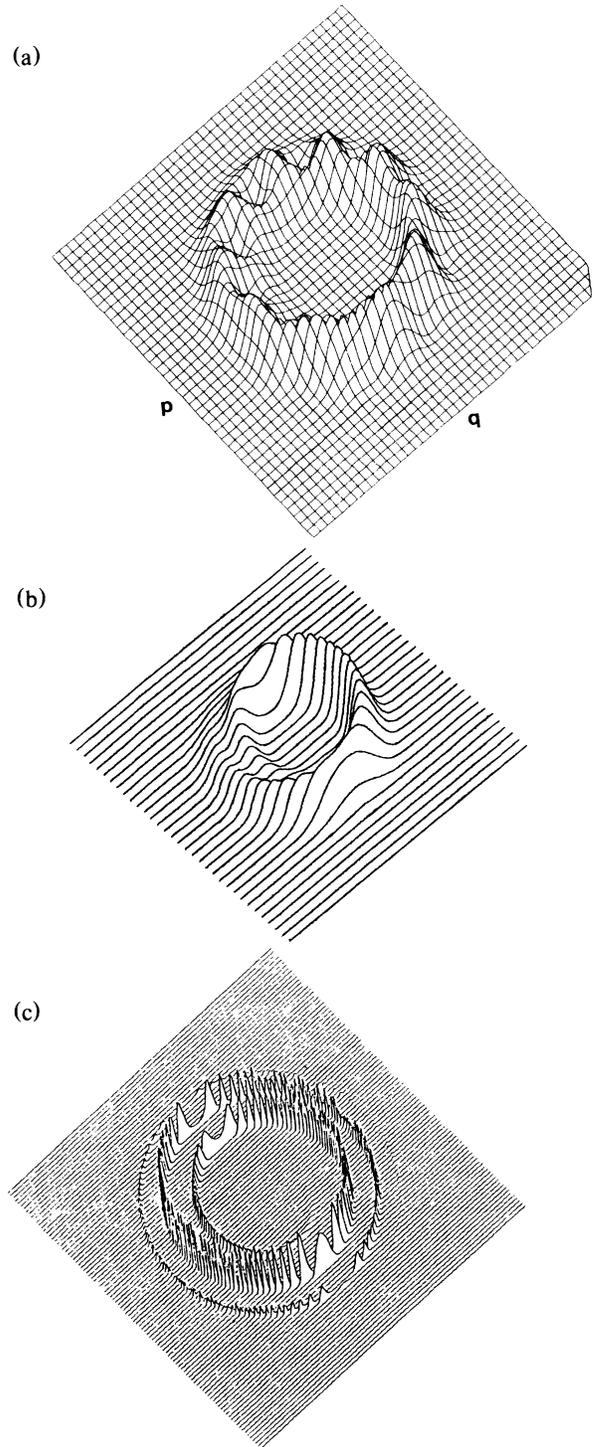

FIG. 1. Plot of Q-function probability density on phase space. The origin is in the center of each figure. (a) Quantum case $\alpha_0 = 4i$, $\nu = 0.6$, $\kappa = 0.0$; orientation of position (q) and momentum (p) axes is shown. (h) Quantum case $\alpha_0 = 4i$, $\nu = 0.6$, $\kappa = 0.5$. (c) Classical case $\alpha_0 = 4i$, $\nu = 0.6$, $\kappa = 0.0$.

may be realized as an intracavity field made to interact with a nonlinear crystal. This mode is unavoidably coupled to the electromagnetic vacuum through the cavity mirrors even though for a high-Q cavity this coupling will be very small. The essential conclusion of the discussion here is that although the dissipation may be so small as to be almost unobservable as a general contraction of phase-space dynamics, it may be sufficient to induce a reduction of quantum dynamics to classical dynamics as the energy of the system approaches the classical scale. The effectiveness of even small dissipation in suppressing quantum coherences has also been emphasized by Caldeira and Leggett[13] and Zurek.[14] We suggest that such a model may provide a unified description of microscopic and macroscopic dynamics. A similar suggestion has recently been made by Ghirardi, Rimini, and Weber.[15]

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